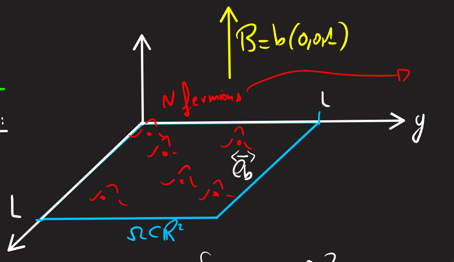
Multiple Landau level filling for a large magnetic field limit of 2D fermions

Model:

Context:



- in Quantum mechanics: state $\in$ Hilbert (points $\to$ densities)
- physical quantities (observables) $\to$ operators on Hilbert
- result of measure $\to$ eigen-value

$$\Psi_N \in L^2(\Omega^N) \;\Big|\; \begin{cases} \int_{\Omega^N} |\Psi_N(x_{1:N})|^2 \, dx_{1:N} = 1 \\ \forall \sigma \in S_N,\ \Psi_N(x_{\sigma(1)}, \dots, x_{\sigma(N)}) = \varepsilon(\sigma)\Psi_N(x_{1:N}) \end{cases}$$

$\Psi_N \in \bigwedge_{i=1}^{N} L^2(\Omega) = \mathrm{span}\{\phi_1 \wedge \dots \wedge \phi_N,\ \phi_{1:N} \in L^2(\Omega)\}$,

Pauli principle: $\phi \wedge \phi = 0$

Goal: energy and density at $T = 0K$ (Ground state) when $N, b \to +\infty$

1 body kinetic operator:

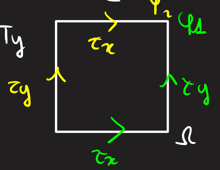
$$\mathcal{L} := (-i\hbar\nabla - bA)^2,\ (m=1, c=1, q=1),\quad \nabla \wedge A = (0,0,1),\ \text{Coulomb gauge: } \nabla . A = 0 \text{ or } A = \nabla^\perp \phi = \begin{pmatrix} -\partial_y \phi \\ \partial_x \phi \end{pmatrix},\ \underline{\text{ex}}: A_{\text{Landau}}(x, y) \begin{pmatrix} -y \\ 0 \end{pmatrix}$$

- magnetic periodic boundary conditions: $P := -i\hbar\nabla - bA$, $z \in \Omega$, $T_z\Psi := \Psi(\cdot - z)$, $[P, T_z] = PT_z - T_zP = (-bA + bA(\cdot - z))T_z \neq 0$

$\hookrightarrow$ Translation symmetry broken by the magnetic fied, but $\nabla \wedge (A - A(\cdot - z)) = 0$ so $bA - bA(\cdot - z) = \hbar\nabla\varphi_z$

magnetic translation: $\tau_z := e^{i\varphi_z} T_z$, $[P, \tau_z] = [-i\hbar\nabla - bA, e^{i\varphi_z} T_z] = (\hbar\nabla\varphi_z - bA + A(\cdot - z))\tau_z = 0$

$[\tau_x, \tau_y] = (e^{i\varphi_1} - e^{i\varphi_2}) T_x T_y$



$, \varphi_1 - \varphi_2 = \frac{1}{\ell_b^2} \int_{\partial\Omega} A . dl = \frac{L^2}{\ell_b^2}$ so $[\tau_x, \tau_y] = 0 \iff \frac{L^2}{\ell_b^2} = 2\pi d$ (F.Q)

Stocke's theorem, where ℓ_b = hbar/b

$H^k_{mp}(\Omega) := \{\Psi|_\Omega,\ \Psi \in H^2_{loc}(\mathbb{R}^2) \mid \tau_x\Psi = \tau_y\Psi = \Psi\}$, $\mathrm{Dom}(\mathcal{L}) := H^2_{mp}(\Omega)$

- Landau levels: $\mathcal{L} =: \sum_{n \in \mathbb{N}} E_n \Pi_n$ (rank d projection), $nLL := \Pi_n \mathrm{Dom}(\mathcal{L})$, $E_n =: 2\hbar b(n + \frac{1}{2})$, $\sum_{n \in \mathbb{N}} \Pi_n = \mathrm{Id}_{L^2(\Omega)}$

N body operator:

$$\mathcal{H}_N := \sum_{i=1}^{N} \big(\mathcal{L}(x_i) + \underbrace{V(x_i)}_{\text{external potential}}\big) + \underbrace{\frac{2}{N-1}}_{\text{mean field scaling}} \overbrace{\sum_{i<j} \underbrace{w(x_i - x_j)}_{\text{interaction potential}}}^{\frac{N(N-1)}{2} \text{ terms}},\quad \mathrm{Dom}(\mathcal{H}_N) = \bigwedge_{i=1}^{N} \mathrm{Dom}(\mathcal{L})$$

Scaling: mean density: $\frac{L^2}{N}$, $\begin{cases} \frac{L}{\sqrt{N}} : \text{mean distance between particles} \\ \ell_b : \text{minimal distance between particles (Pauli principle)} \end{cases}$ $\left(\begin{array}{l} \cdot \text{ classically radius of cyclotron orbit} \\ \cdot \text{ eigenvector of } \mathcal{L}:\ \Psi(z) = e^{-\left|\frac{z}{\ell_b}\right|^2} \end{array} \right)$

$$\nu := \left(\frac{\ell_b}{\frac{L}{\sqrt{N}}}\right)^2 = \frac{\hbar N}{bL^2}$$

- Lieb-Solovej-Yngvason '95 $\nu \to 0$ in $\mathbb{R}^3$ with $w(x - y) = \frac{1}{|x - y|}$ $\begin{cases} \text{all particles in } 0LL \\ \text{electrostatic model} \quad \rho \mapsto \int (V + w * \rho)\rho \end{cases}$

- $\nu \to +\infty$ all LL filled: Thomas-Fermi model: $\rho \mapsto \int \rho^{1 + \frac{2}{d}} + \int (V + w * \rho)\rho$

- Fournais-Lewin-Solovej "15, $\mathbb{R}^d$
- Fournais-Madsen "19, $\mathbb{R}^3$

) Magnetic TF with general V, w $\quad \rho \mapsto \underbrace{\int \mathcal{G}(\rho)}_{\text{depends on the scaling}} + \int (V + w * \rho)\rho$

Quantum Hall effect: $0LL, \dots, (q-1)LL$ filled, qLL partially filled with $\overbrace{\text{filling factor } r}^{\text{charge carriers in qLL}}$.

fix $\frac{N}{d} \to q + r$, $q \in \mathbb{N}$, $r \in [0,1)$, so $\nu = \frac{N\ell_b^2}{L^2} \underset{(F.Q)}{=} \frac{N}{2\pi d} \to \frac{q + r}{2\pi}$, Ω bounded $\Rightarrow$ $\underbrace{d \text{ finite}}_{\text{necessary}}$, semi-classical: $\hbar := N^{-\delta}$

$\Downarrow$

$\ell_b = O(N^{-1/2})$, $b = \frac{\hbar}{\ell_b^2} = O(N^{1-\delta})$, kinetic energy: $\hbar b = O(N^{1-2\delta}) \gg 1$, take $\overbrace{\frac{1}{4} < \delta}^{\text{technical reason}} < \underbrace{\frac{1}{2}}_{1-2\delta > 0}$

Semiclassical model:

Semiclassical density: $m : (\mathbb{N}\times\Omega, \underbrace{\eta := \sum_{n\in\mathbb{N}} \delta_n \otimes \mathrm{Leb}_{|\Omega}}_{\text{phase-space}}) \to \mathbb{R}_+ \mid m \leq \frac{1}{L^2(q+r)}$ (Pauli), $\int m\, d\eta = \sum_{n\in\mathbb{N}} \int_\Omega m(n,x)\,dx = 1$

$$\mathcal{E}_{x,N}(m) = \int_{\mathbb{N}\times\Omega} (E_n + V(x))\, m(n,x)\, d\eta(n,x) + \int_{(\mathbb{N}\times\Omega)^2} w(x-y)\, m(n,x)\, m(\tilde{n}, y)\, d\eta(n,x)\, d\eta(\tilde{n}, y)$$

Electrostatic model for qLL: $\mathcal{D}_{qLL} = \left\{ \rho \in L^1(\Omega) \,\middle|\, \underbrace{\int_\Omega \rho\, dx = \frac{r}{q+r}}_{\text{proportion of particles in qLL}},\ \underbrace{0 \leq \rho \leq \frac{1}{L^2(q+r)}}_{\text{Pauli principle}} \right\}$, $\mathcal{E}_{qLL}(\rho) := \int (V + w * \rho)\rho$

Let $\rho \in \mathcal{D}_{qLL}$, $m_\rho(n,x) := \frac{1}{L^2(q+r)} \mathbb{1}_{n<q} + \rho(x)\mathbb{1}_{n=q}$, then $\mathcal{E}_{x,N}(\rho) =: \underbrace{\hbar b E^{q,r}}_{\text{kinetic energy in q lowest LL}} + \underbrace{E_V^{q,r}}_{\text{potential energy in q lowest LL}} + \underbrace{E_w^{q,r}}_{\text{interaction energy except those in qLL}} + \mathcal{E}_{qLL}(\rho)$

integrate to one --> correct Pauli bound

Results:

Theorem: (Ground state energy convergence)

With all notations defined above, if V, w in L²(omega)

$$\min_{\substack{\Psi_N \in \mathcal{D}om(\mathcal{H}_N) \\ \|\Psi_N\|_{L^2} = 1}} \frac{\langle \Psi_N | \mathcal{H}_N \Psi_N \rangle}{N} \underset{N\to+\infty}{=} \inf_{\rho\in\mathcal{D}_{qLL}} \mathcal{E}_{x,N}[m_\rho] + o(1)$$

Reduced densities:

Let $\gamma_N \in \mathcal{L}^1(L^2_-(\Omega^N))$, $0 \leq \gamma_N$, $\mathrm{Tr}[\gamma_N] = 1$ (density matrix)

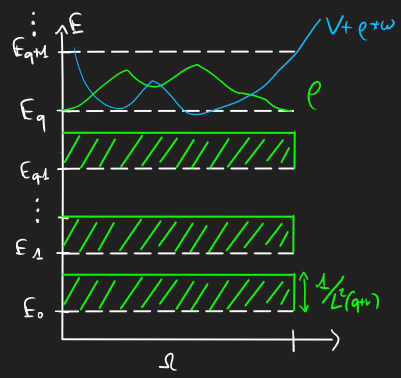
$\gamma_N^{(k)} := \mathrm{Tr}_{k+1:N}[\gamma_N]$, $\left(A, B \in \mathcal{L}^1(\Omega),\ A\otimes B \in \mathcal{L}^1(\Omega^2),\ \mathrm{Tr}_1[A\otimes B] = \mathrm{Tr}[A]\, B\right)$

$\gamma_N^{(k)}(x_{1:k}, y_{1:k}) = \int_{\Omega^{N-k}} \gamma_N(x_{1:k}, z_{k+1:N}; y_{1:k}, z_{k+1:N})\, dz_{k+1:N}$, $\rho_N^{(k)}(x_{1:k}) := \gamma_N^{(k)}(x_{1:k}, x_{1:k})$ (marginals)

Theorem: (Ground state density convergence) if V, w in L²(omega)

Let $\Psi_N := \underset{\substack{\Psi_N \in \mathcal{D}om(\mathcal{H}_N) \\ \|\Psi_N\|_2 = 1}}{\mathrm{argmin}} \frac{\langle \Psi_N | \mathcal{H}_N \Psi_N\rangle}{N}$, With all notations defined above, $\exists \mu \in \mathcal{P}(\mathcal{D}_{qLL})$ such that:

- μ only charges minimizers of $\mathcal{E}_{qLL}$
- in the sense of Radon measures: $\forall k \in \mathbb{N}^*$, $\rho_N^{(k)} \underset{N\to+\infty}{\overset{*}{\rightharpoonup}} \int_{\mathcal{D}_{qLL}} \left(\frac{q}{L^2(q+r)} + \rho\right)^{\otimes k} d\mu(\rho)$



Tools and sketch of the proof:

Quantization: . Landau levels: $P = -i\hbar\nabla - bA =: \begin{pmatrix} p_x \\ p_y \end{pmatrix}$, $a = \frac{p_x + ip_y}{\sqrt{2\hbar b}}$, $a^\dagger = \frac{p_x - ip_y}{\sqrt{2\hbar b}}$, $\mathcal{N} = a^\dagger a$

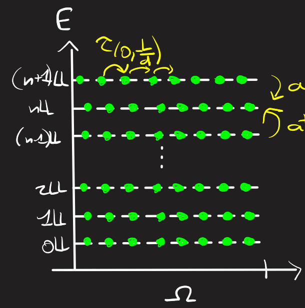
we have: $[a, a^\dagger] = \mathrm{Id}$, $\mathcal{L} = 2\hbar b\left(\mathcal{N} + \frac{1}{2}\right)$, $sp(\mathcal{N}) = \mathbb{N}$, $n\mathrm{LL} = \{\Psi \in \mathcal{D}om(\mathcal{L}) \mid \mathcal{N}\Psi = n\Psi\}$

. lowest LL: $\to 0\mathrm{LL} \subset \mathcal{O}(\Omega)\, e^{-\frac{\phi}{\ell_b^2}}$

$\to f e^{-\frac{\phi}{\ell_b^2}} \in 0\mathrm{LL} \Rightarrow f$ has d zeros inside Ω

$\to \dim(0\mathrm{LL}) = d$ (d independent Fourier coefficients)

In Landau gauge: $\Psi_{0\ell}(x,y) = \frac{1}{\pi^{1/4}\sqrt{L\ell_b}} \sum_{k\in\mathbb{Z}} e^{2i\pi k d\frac{x}{L} - \frac{1}{2\ell_b^2}(y+kL)^2}$, $\Psi_{n\ell} = \frac{(a^\dagger)^n}{\sqrt{n!}} \left(\tau_{(0,-\frac{L}{d})}\right)^\ell \Psi_{00}$



Projectors: $\Pi_n = \sum_{\ell=0}^{d-1} |\Psi_{n\ell}\rangle\langle\Psi_{n\ell}|$, $g_\lambda^2 \underset{\lambda\to 0}{\to} \delta$, $\sqrt{N}\hbar^2 \ll \lambda \ll 1$

localization: $\underbrace{\Pi_{n,x}}_{\text{projection on phase space}} = g_\lambda(\cdot - x)\, \Pi_n\, g_\lambda(\cdot - x)$,

prop: $\Pi_n \underset{N\to+\infty}{\sim} \Pi_n^\infty$, $\int_{\mathbb{N}\times\Omega} \Pi_x\, d\eta(x) = \mathrm{Id}$

Semi-classical limit: Husimi fonctions: $m^{(k)}_{\gamma_N}(X_{1:k}) := \mathrm{Tr}\left[\gamma_N^{(k)} \bigotimes_{i=1}^{k} \Pi_{X_i}\right]$, Pauli : m^(k) \le 1/(L^2(q+r))^k + o(1)

$$\mathcal{E}_{sc}(m^{(1)}, m^{(2)}) := \int_{\mathbb{N}\times\Omega} (E_n + V(x))\, m^{(1)}(n,x)\, d\eta(n,x) + \int_{(\mathbb{N}\times\Omega)^2} w(x-y)\, m^{(2)}(n,x;\tilde{n},y)\, d\eta(n,x)\, d\eta(\tilde{n},y)$$

Prop: $\dfrac{\mathrm{Tr}[\mathcal{H}_N \gamma_N]}{N} = \mathrm{Tr}\left[(\mathcal{L}+V)\gamma_N^{(1)}\right] + \mathrm{Tr}\left[w\, \gamma_N^{(2)}\right] = \mathcal{E}_{sc}(m_N) + o(1)$, for a density matrix gamma_N

Mean field limit: Control the correlations beyond the correlations due to antisymmetry

If $m^{(2)} = m^{(1)} \otimes m^{(1)}$ then $\mathcal{E}_{sc}(m^{(1)}, m^{(2)}) = \mathcal{E}_{sc}(m^{(1)})$

Upper bound: Hartree-Fock theory, restrict to slater states: $\Psi_N = \frac{1}{\sqrt{N!}} \bigwedge_{i=1}^{N} \phi_i$, with $\phi_{1:N}$ an orthonormal family of $L^2(\Omega)$

Theorem: Wick gamma_N = proj on Psi_N

$$\gamma_N^{(1)}(x,y) = \frac{1}{N} \sum_{i=1}^{N} \phi_i(x)\overline{\phi_i(y)}, \quad \gamma_N^{(2)}(x_1,x_2; y_1,y_2) = \frac{N}{N-1}\Big(\underbrace{\gamma_N^{(1)}(x_1,y_1)\gamma_N^{(1)}(x_2,y_2)}_{\text{direct}} - \underbrace{\gamma_N^{(1)}(x_1,y_2)\gamma_N^{(1)}(x_2,y_1)}_{\text{exchange}}\Big)$$

Theorem: Lieb's variational principle

Let $\gamma \in \mathcal{L}^1(L^2(\Omega))$, $0 \le \gamma \le \frac{1}{N}$, $\mathrm{Tr}[\gamma] = 1$, define γ_2 with Wick's formula then:
$\exists \gamma_N \in \mathcal{L}^1(L^2_-(\Omega^N)), \gamma_N \ge 0,\ L_2 \in \mathcal{L}^1(L^2(\Omega)), L_2 \ge 0 \mid \gamma_N^{(1)} = \gamma, \gamma_N^{(2)} = \gamma_2 - L_2$

For $\rho \in \mathcal{D}_{qLL}$, define $\gamma := 2\pi\ell_b^2 \int m_\rho(X)\Pi_X\, d\eta(X)$, Prop: $0 \le \gamma \le \frac{1}{N}$, $\mathrm{Tr}[\gamma] = 1 + o(1)$

after small modifications on m_ρ, Lieb's variational principle applies:

$$\frac{\mathrm{Tr}[\mathcal{H}_N\gamma_N]}{N} = \mathrm{Tr}[(\mathcal{L}+V)\gamma] + \mathrm{Tr}[w(\gamma_2 - L_2)] \underset{L_2 \ge 0}{\le} \underbrace{\mathrm{Tr}[(\mathcal{L}+V)\gamma + \mathrm{Tr}[w\gamma_2]}_{:= \mathcal{E}^{HF}(\gamma)} + o(1) = \mathcal{E}_{sc}(m_\rho) + o(1)$$

Control exchanges terms
$\mathrm{Tr}[\gamma_{ex}] = \mathrm{Tr}[\gamma^2] \le \frac{1}{N}$

Lower bound:

Theorem: De Finetti

Let $\mu \in \mathcal{P}_S(\Omega^{\mathbb{N}})$ with marginals $(\mu_n)_{n\in\mathbb{N}^*}$, $\exists P_\mu \in \mathcal{P}(\mathcal{P}(\Omega)) \mid \forall n \in \mathbb{N}^*, \mu_n = \int_{\mathcal{P}(\Omega)} \rho^{\otimes n}\, dP_\mu(\rho)$

Take: $\Psi_N := \underset{\substack{\Psi_N \in \mathrm{Dom}(\mathcal{H}_N) \\ \|\Psi_N\|_2 = 1}}{\mathrm{argmin}} \dfrac{\langle \Psi_N | \mathcal{H}_N \Psi_N\rangle}{N}$, after extraction $m^{(k)}_{\Psi_N} \overset{*}{\rightharpoonup} M^{(k)}$ on $L^\infty((\mathbb{N}\times\Omega)^k)$, De Finetti $M^{(k)} = \int_{\mathcal{P}(\mathbb{N}\times\Omega)} m^{\otimes k}\, dP_M$

M^(1) same

Prop: P_M a.e. $m(q,\cdot) \in \mathcal{D}_{qLL}$ and $m(n,x) = \frac{1}{L^2(q+r)} \mathbb{1}_{n<q} + m(q,x)\,\mathbb{1}_{n=q}$

So $\mathcal{E}_{sc}(m^{(1)}_{\Psi_N}, m^{(2)}_{\Psi_N}) \underset{N\to+\infty}{\to} \mathcal{E}_{sc}(M^{(1)}_{\Psi_N}, M^{(2)}_{\Psi_N}) = \int_{\mathcal{P}(\mathbb{N}\times\Omega)} \mathcal{E}_{sc}(m, m\otimes m)\, dP_M = \int_{\mathcal{P}(\mathbb{N}\times\Omega)} \mathcal{E}_{sc}(m)\, dP_M \ge \inf_{\rho\in\mathcal{D}_{qLL}} \mathcal{E}_{sc}(m_\rho)$

(Lieb-Thirring inequality densities $\in L^2$) linearity

$\mu = (m \mapsto m(q,\cdot))_* P_M \in \mathcal{P}(\mathcal{D}_{qLL})$ upper bound $\Rightarrow$ μ only supported on minimizers